INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids 2005; **47**:955–962
Published online 24 January 2005 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/fld.906

Numerical modelling of flow with heat transfer in petroleum reservoirs

M. Amara[1], D. Capatina[1], B. Denel[1,*,†] and P. Terpolilli[2]

[1] *Université de Pau, L.M.A. CNRS-FRE 2570, Avenue de l'Université, 64000 Pau, France*
[2] *Total, CST Jean Feger, Avenue Larribau, 64018 Pau Cedex, France*

SUMMARY

We model and analyse the single-phase flow in a petroleum reservoir by taking into account a non-standard energy equation. The numerical approximation is based on Raviart–Thomas mixed finite elements and a code is developed and validated. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: petroleum reservoir; thermometrics; porous medium; mixed finite elements

1. INTRODUCTION

Thanks to emerging technologies based on optical fibre, some petroleum wellbores are now endowed with temperature sensors. In order to interpret the recorded temperature profiles and to predict flow repartition among each producing layer of a reservoir or to estimate virgin reservoir temperatures, one first needs to develop a forward model. This implies to couple a reservoir and a wellbore model, respectively, describing the flow of a compressible fluid (oil or gas) from both a dynamic and a thermal point of view, in a porous and a fluid medium.

In this paper, we are interested in the modelling and the finite element approximation of a reservoir. The reservoir Ω is treated as a porous medium divided into several geological layers $(\Omega_i)_{1\leqslant i\leqslant N}$ which are characterized by their own dip and physical properties. The fluid flow is modelled by the Darcy–Forchheimer equation coupled with a non-standard energy balance which takes into account, besides convection and diffusion, the compressibility effect (Joule–Thomson effect) and the frictional heating that occurs in the formation.

The physical problem is first written in axisymmetric form and then time-discretized. Thus, at each time step one gets a linear system which is shown to be well-posed.

*Correspondence to: B. Denel, Laboratoire de Mathématiques Appliquées, Université de Pau et des Pays de l'Adour, IPRA—Avenue de l'Université, BP 1155 64013 Pau Cedex, France.
†E-mail: bertrand.denel@univ-pau.fr

Contract/grant sponsor: Total; contract/grant number: DGEP/TDO/CA/RD No. 12698

Received 27 April 2004
Revised 7 December 2004
Accepted 9 December 2004

The numerical approximation is achieved by means of conservative Raviart–Thomas finite elements. The existence and uniqueness of the discrete solution is established thanks to a variant of the Babuska–Brezzi theory for mixed variational formulations. Numerical tests are presented, validating the model from both a numerical (convergence in time and space) and a physical (comparison with analytical solutions for the pressure) point of view.

2. PHYSICAL MODELLING

Due to the geometry of the domain (a reservoir surrounding a cylindrical well), we write our problem in 2D axisymmetric form, depending only on the cylindrical coordinates (r, z). Following the ideas of Ewing *et al.* [1], the pressure and the temperature are taken independent of θ and our 2D domain consists merely of $\Omega = \{(r, z) \mid r_w \leqslant r \leqslant R, z \in [z_{\min}, z_{\max}]\}$. The (r, z) formulation of the conservation laws describing our problem, after a change of variables is

$$
\begin{gathered}
r\phi \frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{G} = 0, \quad \frac{1}{r\rho}\left(\mu \underline{\mathbf{K}}^{-1} + \frac{F}{r}|\mathbf{G}|\underline{\mathbf{I}}\right)\mathbf{G} + \nabla p = -\rho \mathbf{g} \\
r(\rho c)_* \frac{\partial T}{\partial t} + \rho^{-1}(\rho c)_f \mathbf{G} \cdot \nabla T - \operatorname{div} \mathbf{q} - r\phi\beta T \frac{\partial p}{\partial t} - \rho^{-1}(\beta T - 1)\mathbf{G} \cdot \nabla p = 0 \\
\frac{1}{r\lambda}\mathbf{q} - \nabla T = 0 \quad \rho = \rho(p, T)
\end{gathered}
\tag{1}
$$

One notes that (1) is a coupled nonlinear system composed of the mass conservation law, the Darcy–Forchheimer equation, a non-standard energy equation and the cubic state equation of Peng–Robinson. The unknowns are the specific flux $\mathbf{G} = r\rho\mathbf{v} = (G_r, G_z)^t$ with $\mathbf{v}$ denoting the Darcy's velocity, the heat flux $\mathbf{q} = (q_r, q_z)^t$, the pressure p, the temperature T and the density ρ. The other coefficients are thermal or physical ones which can be discontinuous across the interfaces of the geological layers.

We add to (1) initial conditions for ρ and T and boundary conditions. For that, we take $\partial\Omega = \overline{\Gamma}_G \cup \overline{\Gamma}_p = \overline{\Gamma}_q \cup \overline{\Gamma}_T$ and we impose a pressure p_Γ, a temperature T_Γ, a normal specific flux Q and finally a normal heat flux Ψ on Γ_p, Γ_T, Γ_G and Γ_q, respectively.

3. TIME-DISCRETIZED PROBLEM

The time discretization is based on the Euler's implicit scheme. At each time loop, the idea is to determine the unknowns $\mathbf{G}$, $\mathbf{q}$, p and T and then to update ρ by verifying the Peng–Robinson cubic equation. To do that, we replace the derivative of ρ thanks to its dependency in both p and T, and we obtain the following linearized problem:

$$
\begin{gathered}
\frac{1}{r}\underline{\mathbf{M}}\mathbf{G} + \nabla p = -\rho^{n-1}\mathbf{g}, \quad \frac{1}{r\lambda}\mathbf{q} - \nabla T = 0 \\
r\frac{a}{\Delta t}p - r\frac{b}{\Delta t}T + \operatorname{div}\mathbf{G} = r\frac{a}{\Delta t}p^{n-1} - r\frac{b}{\Delta t}T^{n-1} \\
r\frac{d}{\Delta t}T + k\mathbf{G}^{n-1} \cdot \nabla T - r\frac{f}{\Delta t}p + l\mathbf{G}^{n-1} \cdot \nabla p - \operatorname{div}\mathbf{q} = r\frac{d}{\Delta t}T^{n-1} - r\frac{f}{\Delta t}p^{n-1}
\end{gathered}
\tag{2}
$$

where a, b, d, k, f, l and $\underline{\mathbf{M}}$ refer to coefficients computed at t^{n-1}.

In practice $r_w \simeq 4$ inch, so we are not concerned here with the case where r goes to 0, see also Reference [1].

The present reservoir model is coupled in Reference [2] with a wellbore model, where now r may vanish. This degeneracy is treated by developing a pseudo 1D approach, thanks to an explicit dependence on r of the unknowns.

For the sake of simplicity, we denote by $\mathbf{V} = (\mathbf{G}, \mathbf{q})$ the vector unknowns, respectively, by $s = (p, T)$ the scalar ones and we introduce the spaces

$$\mathbb{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega), \quad \mathbb{H}(\mathrm{div}, \Omega) = H(\mathrm{div}, \Omega) \times H(\mathrm{div}, \Omega)$$

$$\mathbb{H}^0(\mathrm{div}, \Omega) = \{\mathbf{V}' = (\mathbf{G}', \mathbf{q}') \in \mathbb{H}(\mathrm{div}, \Omega); \mathbf{G}' \cdot \mathbf{n} = 0 \text{ on } \Gamma_G, \ \mathbf{q}' \cdot \mathbf{n} = 0 \text{ on } \Gamma_q\}$$

$$\mathbb{H}^*(\mathrm{div}, \Omega) = \{\mathbf{V}' = (\mathbf{G}', \mathbf{q}') \in \mathbb{H}(\mathrm{div}, \Omega); \mathbf{G}' \cdot \mathbf{n} = Q \text{ on } \Gamma_G, \ \mathbf{q}' \cdot \mathbf{n} = \Psi \text{ on } \Gamma_q\}$$

We establish next the well-posedness of the time-discretized problem (2) at any t^n, under non-restrictive regularity assumptions on the data but for a sufficiently small Δt. In order to do that, we neglect in a first time the convective terms and we write the problem under a mixed variational form

$$\begin{aligned} &\text{Find } \mathbf{V} \in \mathbb{H}^*(\mathrm{div}, \Omega), \quad s \in \mathbb{L}^2(\Omega) \text{ such that} \\ &A(\mathbf{V}, \mathbf{V}') + B(s, \mathbf{V}') = F_1(\mathbf{V}') \quad \forall \mathbf{V}' \in \mathbb{H}^0(\mathrm{div}, \Omega) \\ &-B(s', \mathbf{V}) + C(s, s') = F_2(s') \quad \forall s' \in \mathbb{L}^2(\Omega) \end{aligned} \tag{3}$$

where the bilinear forms are defined by

$$A(\mathbf{V}, \mathbf{V}') = \int_\Omega \frac{1}{r} \underline{\mathbf{M}}\mathbf{G} \cdot \mathbf{G}' \, \mathrm{d}x + \int_\Omega \frac{1}{r\lambda} \mathbf{q} \cdot \mathbf{q}' \, \mathrm{d}x B(s, \mathbf{V}') = -\int_\Omega p \, \mathrm{div} \, \mathbf{G}' \, \mathrm{d}x + \int_\Omega T \, \mathrm{div} \, \mathbf{q}' \, \mathrm{d}x$$

$$C(s, s') = \int_\Omega r \frac{a}{\Delta t} p p' \, \mathrm{d}x - \int_\Omega r \frac{b}{\Delta t} T p' \, \mathrm{d}x + \int_\Omega r \frac{d}{\Delta t} T T' \, \mathrm{d}x - \int_\Omega r \frac{f}{\Delta t} p T' \, \mathrm{d}x$$

Then one can establish:

Theorem 3.1

Assume that all the thermodynamic coefficients are bounded and that a, d, $1/\lambda$ are strictly positive and $\underline{\mathbf{M}}$ is uniformly positive definite. Suppose moreover that

$$\exists c \in R_+^* \text{ such that } 4ad - (b + f)^2 \geqslant c \text{ a.e. in } \Omega \tag{4}$$

Then the mixed problem (3) has a unique solution.

Proof

We apply a variant of the Babuska–Brezzi theory. One easily checks the coercivity of $A(\cdot, \cdot)$ on $\mathrm{Ker}\, B$ and the inf–sup condition for $B(\cdot, \cdot)$. Moreover, $A(\cdot, \cdot)$ is obviously symmetric and positive on $\mathbb{H}(\mathrm{div}, \Omega)$. Hence, it is sufficient (cf. Reference [3]) to prove the positivity of $C(\cdot, \cdot)$, which in our case is ensured by the condition (4). For more details, one may see Reference [4]. Note that (4) is justified in practice by all the available experimental data. □

One can also show that for sufficiently smooth boundary conditions and thermodynamic coefficients (i.e. $\nabla\lambda \in L^\infty(\Omega_i), \underline{\mathbf{M}}^{-1} \in \mathscr{C}^{0,1}(\overline{\Omega_i})$ Lipschitz function), the solution to (3) is smoother on each geological layer Ω_i. More precisely, one gets that $s \in \mathbb{Y} = \prod_{i=1}^{N} \mathbb{H}^{1+\delta}(\Omega_i)$ with $0 < \delta \leqslant 1$. The proof is based on the regularity of an elliptic problem with discontinuous coefficients on a polygon (cf. Reference [5], see also Reference [4]). An important point is that we conserve the regularity of the solution from one time step to another. Therefore, the linear continuous operator $K : \mathbb{Y} \longrightarrow L^2(\Omega)$, $K(s) = k\mathbf{G}^{n-1} \cdot \nabla T + l\mathbf{G}^{n-1} \cdot \nabla p$ is compact thanks to the compact embedding of $H^\delta(\Omega_i)$ into $L^2(\Omega_i)$.

Denoting by $\mathscr{L}$ the linear continuous operator which associates with any smooth data $\mathbf{f}$ the unique solution $\sigma = (\mathbf{V}, s) \in \mathbb{H}(\mathrm{div}, \Omega) \times \mathbb{Y}$ to (3), our initial problem (2) can be put under the following form:

$$\sigma = \mathscr{L}(\mathbf{f} + \mathscr{K}(\sigma)) \iff (\mathscr{I} - \mathscr{L}\mathscr{K})\sigma = \mathscr{L}\mathbf{f} \tag{5}$$

where now $\mathscr{L}\mathscr{K}$ is a compact operator from $\mathbb{H}(\mathrm{div}, \Omega) \times \mathbb{Y}$ to itself.

Theorem 3.2
For Δt sufficiently small, problem (5) has a unique solution.

Proof
Thanks to the Fredholm's alternative, it is sufficient to prove that $\mathrm{Ker}(\mathscr{I} - \mathscr{L}\mathscr{K}) = \{0\}$ in order to obtain the well-posedness of (5). The solution of the equation $(\mathscr{I} - \mathscr{L}\mathscr{K})\sigma = 0$ satisfies the following relation:

$$A(\mathbf{V}, \mathbf{V}) + C(s, s) + \int_\Omega K(s) T \, \mathrm{d}x = 0$$

By replacing $\nabla T = (1/r\lambda)\mathbf{q}$ and $\nabla p = -(1/r)\underline{\mathbf{M}}\mathbf{G}$, it finally comes that $\sigma = 0$ for $\Delta t < c'$ $[4ad - (b + f)^2]$ where c' is a constant depending on $\underline{\mathbf{M}}$ and $\mathbf{G}^{n-1}$. A more detailed proof can be found in Reference [4]. □

As a conclusion, we have proved that at any time step t^n the linearized problem (2) has a unique solution. The well-posedness of the non-linear time continuous problem (1) is to be studied in perspective.

4. FINITE ELEMENT APPROXIMATION

Let $(\mathscr{T}_h)_h$ be a regular family of triangulations of $\overline{\Omega}$ consisting of triangles matching at the interfaces between the layers Ω_i. We consider the following conforming finite element spaces:

$$L_h = \{p' \in L^2(\Omega);\ p'_{|_K} \in P_0,\ \forall K \in \mathscr{T}_h\}, \quad V_h = \{\mathbf{G}' \in H(\mathrm{div}, \Omega);\ \mathbf{G}'_{|_K} \in RT_0,\ \forall K \in \mathscr{T}_h\}$$

where P_0 is the space of constant functions and RT_0 is the lowest-order Raviart–Thomas space (see Reference [3]). We put

$$\mathbb{L}_h = L_h \times L_h, \quad \mathbb{V}_h^0 = (V_h \times V_h) \cap \mathbb{H}^0(\mathrm{div}, \Omega)$$

$$\mathbb{V}_h^* = \{(\mathbf{G}', \mathbf{q}') \in V_h \times V_h \mid \mathbf{G}' \cdot \mathbf{n} = \mathscr{I}_h(Q) \text{ on } \Gamma_G,\ \mathbf{q}' \cdot \mathbf{n} = \mathscr{I}_h(\Psi) \text{ on } \Gamma_q\}$$

where $\mathscr{I}_h(Q)$, $\mathscr{I}_h(\Psi)$ are piecewise constant approximations of Q on Γ_G, respectively, of Ψ on Γ_q. We employ an upwind scheme in order to treat the convective terms. Thus, we take

$$\int_K k\mathbf{G}_h^{n-1} \cdot \nabla T \,\mathrm{d}x \simeq \sum_{e\in\partial K^-} k\left(\int_e \mathbf{G}_h^{n-1} \cdot \mathbf{n}\,\mathrm{d}\sigma\right)(T^* - T_{|K}), \quad \forall T \in L_h \tag{6}$$

where $\partial K^- = \{e \in \partial K \mid \mathbf{G}_h^{n-1} \cdot \mathbf{n} < 0\}$ is the set of incoming edges. We take $T^* = T_{|K^*}$, where the triangle K^* is such that $\{e\} = \partial K \cap \partial K^*$, $T^* = T_\Gamma$ if $e \in \Gamma_T$, respectively, $T^* = 0$ if $e \in \partial\Omega \backslash \Gamma_T$. A similar formula is used for the pressure term.

We are now able to write the discrete problem

$$\begin{aligned}
&\text{Find } \mathbf{V}_h \in \mathbb{V}_h^*, \quad s_h \in \mathbb{L}_h \text{ such that}\\
&A(\mathbf{V}_h, \mathbf{V}') + B(s_h, \mathbf{V}') = F_{1h}(\mathbf{V}') \quad \forall \mathbf{V}' \in \mathbb{V}_h^0\\
&B(s', \mathbf{V}_h) - (C + D_h)(s_h, s') = F_{2h}(s') + F_{3h}(s') \quad \forall s' \in \mathbb{L}_h
\end{aligned} \tag{7}$$

where $F_{1h}(\cdot)$ and $F_{2h}(\cdot)$ are obtained from $F_1(\cdot)$ and $F_2(\cdot)$ by numerical integration on ρ^{n-1}, p^{n-1}, T^{n-1}. $D_h(\cdot,\cdot)$ comes from the upwinding scheme and $F_{3h}(\cdot)$ too, this last linear form is due to the non-homogeneous boundary conditions.

Concerning the well-posedness of the discrete problem (7), we apply an extension of the Babuska–Brezzi theory which can be found in Reference [3].

Since $\mathrm{Ker}_h B \subset \mathrm{Ker}\, B$, it is obvious that $A(\cdot,\cdot)$ is uniformly $\mathbb{H}(\mathrm{div}, \Omega)$-elliptic. The discrete inf–sup condition on $B(\cdot,\cdot)$ is also satisfied, uniformly with respect to the discretization parameter. The proof is classical and makes use of Fortin's trick and the Raviart–Thomas interpolation operator. Next we state

Lemma 4.1
For $\Delta t \leqslant \alpha h^2/\|\mathbf{G}_h^{n-1}\|_{0,\Omega}$ and with $h \leqslant ch_K$ for all $K \in \mathscr{T}_h$, one has $(C + D_h)(s,s) \geqslant 0$, $\forall s \in \mathbb{L}_h$.

Proof
One uses that $C(s,s) \geqslant (c/\Delta t)\|s\|_{0,\Omega}^2$ and the continuity of $D_h(\cdot,\cdot)$ on $\mathbb{L}_h \times \mathbb{L}_h$

$$\begin{aligned}
|D_h(s,s')| &\leqslant \frac{c}{h}\left(\sum_{e\in\mathscr{E}_h} h_e\|\mathbf{G}_h^{n-1}\cdot\mathbf{n}\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathscr{E}_h} h_e^{-1}\|[s]\|_{0,e}^2\right)^{1/2}\|s'\|_{0,\Omega}\\
&\leqslant \frac{c_1}{h^2}\|\mathbf{G}_h^{n-1}\|_{0,\Omega}\,\|s\|_{0,\Omega}\,\|s'\|_{0,\Omega}
\end{aligned}$$

This last statement holds thanks to the Cauchy–Schwarz inequality and to the equivalence of norms in finite dimensional spaces (see Reference [3]). So the conclusion holds with $\alpha = 2c/c_1$. □

5. NUMERICAL SIMULATIONS

In order to validate the considered model, we first study the behaviour of the solution with respect to the mesh refinement. We consider a reservoir divided into two geological homogeneous layers where only the lower one communicates with the wellbore. We simulate the production of an oil by imposing a difference of pressure between the perforation and the external boundary of the reservoir. The solutions are computed on congruent meshes and we evaluate the error between the solution calculated on the finest mesh (cf. Figure 1) and the ones obtained for the intermediate meshes. We numerically obtain $\|p - p_h\|_{0,\Omega} \leqslant C|h|^{\alpha}$ as error bound for the pressure with $\alpha \simeq 1.46$.

We also compare the computed pressure with analytical pressure solutions given by well-test software such as Pie for two common situations: the evolution of the pressure near the well for a reservoir with a constant pressure boundary and for a closed reservoir. As shown in Figure 2, both methods lead to very similar results. Finally, we consider an existing reservoir

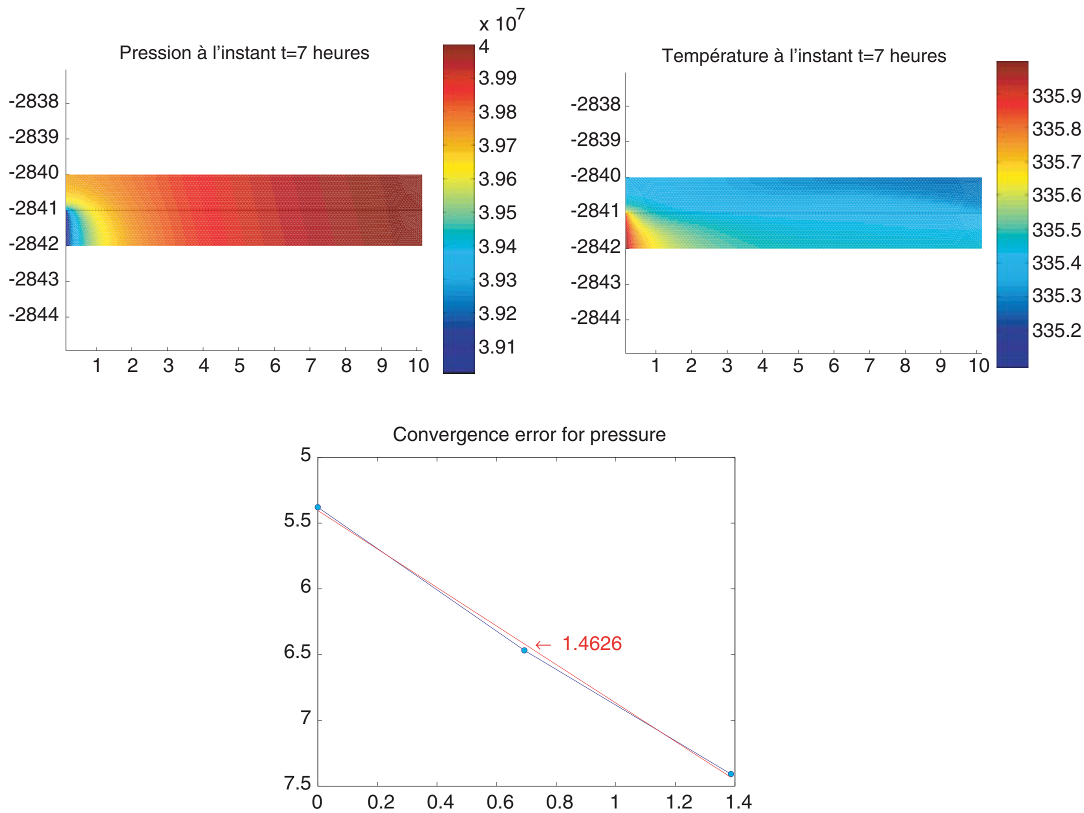

Figure 1. Pressure, temperature and convergence error.

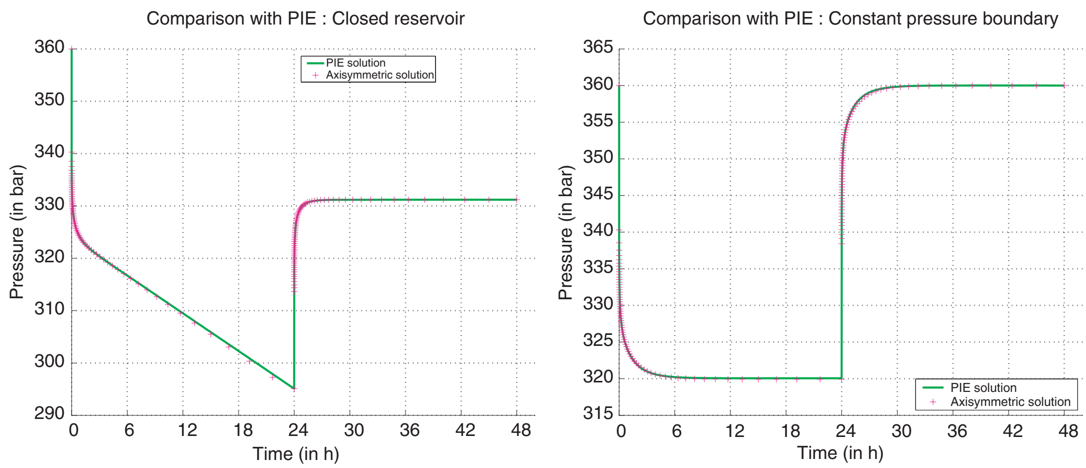

Figure 2. Comparison with analytical pressure solutions for a closed reservoir and a reservoir with a constant pressure boundary.

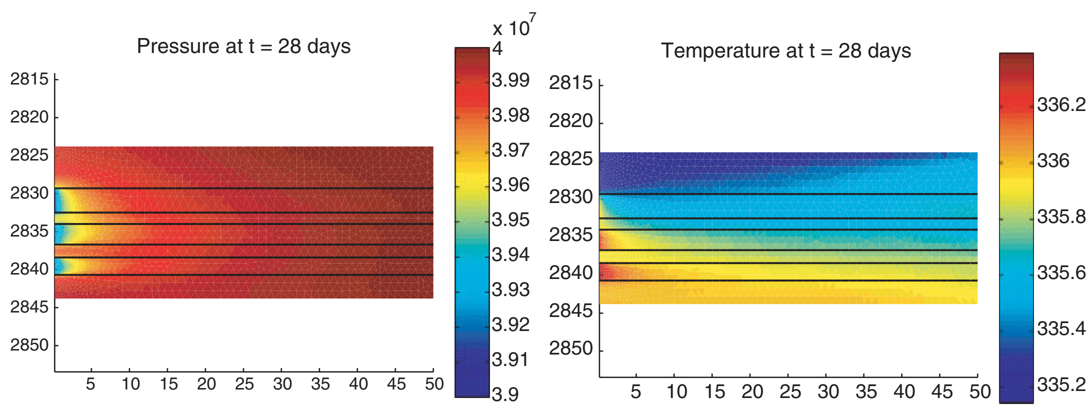

Figure 3. Pressure and temperature maps for a seven-layer realistic reservoir after a one-month production.

divided into seven geological layers characterized by high heterogeneities and where only the even numbered ones communicate with the well. In Figure 3, one may see that the computed pressure and temperature are physically acceptable.

ACKNOWLEDGEMENTS

This work is supported by Total under Contract DGEP/TDO/CA/RD No.12698.

REFERENCES

1. Ewing RE, Wang J, Weekes SL. On the simulation of multicomponent gas flow in porous media. *Applied Numerical Mathematics* 1999; **31**(4):405–427.
2. Denel B. Simulation numérique et couplage de modèles thermomécaniques puits-milieux poreux. *Ph.D. Thesis*, University of Pau, 2004.
3. Roberts JE, Thomas JM. Mixed and hybrid methods. *Handbook of Numerical Analysis*, vol. II. North-Holland: Amsterdam, 1991; 523–639.
4. Amara M, Capatina D, Denel B, Terpolilli P. Modelling, analysis and numerical approximation of flow with heat transfer in a petroleum reservoir. *Preprint No. 0415*, University of Pau, 2004.
5. Grisvard P. *Elliptic Problems on Non-smooth Domains*. Ritman: Boston, 1985.
6. Brezzi F, Fortin M. *Mixed and Hybrid Finite Element Methods*. Springer: New York, 1991.
7. Maubeuge F, Didek M, Arquis E, Bertrand O, Caltagirone JP. A model for interpreting thermometrics. *SPE 28588*, 1994.
8. Peng DY, Robinson DB. A new two-constant equation of state. *Industrial and Engineering Chemistry Fundamentals* 1976; **15**:59–64.

